

ZERO PRODUCT PRESERVERS OF C*-ALGEBRAS

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Dedicated to Professor Bingren Li on the occasion of his 65th birthday (1941.10.7 –)

ABSTRACT. Let $\theta : \mathcal{A} \rightarrow \mathcal{B}$ be a zero-product preserving bounded linear map between C^* -algebras. Here neither \mathcal{A} nor \mathcal{B} is necessarily unital. In this note, we investigate when θ gives rise to a Jordan homomorphism. In particular, we show that \mathcal{A} and \mathcal{B} are isomorphic as Jordan algebras if θ is bijective and sends zero products of self-adjoint elements to zero products. They are isomorphic as C^* -algebras if θ is bijective and preserves the full zero product structure.

1. INTRODUCTION

Let \mathcal{M} and \mathcal{N} be algebras over a field \mathbb{F} and $\theta : \mathcal{M} \rightarrow \mathcal{N}$ a linear map. We say that θ is a *zero-product preserving* map if $\theta(a)\theta(b) = 0$ in \mathcal{N} whenever $ab = 0$ in \mathcal{M} . The canonical form of a linear zero product preserver, $\theta = h\varphi$, arises from an element h in the center of \mathcal{N} and an algebra homomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{N}$. In [6], we see that in many interesting cases zero-product preserving linear maps arise in this way.

We are now interested in the C^* -algebra case. There are 4 different versions of zero products: $ab = 0$, $ab^* = 0$, $a^*b = 0$ and $ab^* = a^*b = 0$. Surprisingly, the original version $ab = 0$ is the least, if any, geometrically meaningful, while the others mean a, b have orthogonal initial spaces, or orthogonal range spaces, or both. Using the orthogonality conditions, the author showed in [11] that a bounded linear map $\theta : \mathcal{A} \rightarrow \mathcal{B}$ between C^* -algebras is a triple homomorphism if and only if θ preserves the fourth disjointness $ab^* = a^*b = 0$ and $\theta^{**}(1)$ is a partial isometry. Here, the triple product of a C^* -algebra is defined by $\{a, b, c\} = (ab^*c + cb^*a)/2$, and $\theta^{**} : \mathcal{A}^{**} \rightarrow \mathcal{B}^{**}$ is the bidual map of θ . See also [3] for a similar result dealing with the case $ab = ba = 0$. We shall deal with the first and original case in this note. The other cases will be dealt with in a subsequent paper.

There is a common starting point of all these 4 versions. Namely, we can consider first the zero products $ab = 0$ of self-adjoint elements a, b in \mathcal{A}_{sa} . In [10] (see also [9]), Wolff shows that if $\theta : \mathcal{A} \rightarrow \mathcal{B}$ is a bounded linear map between unital C^* -algebras preserving the involution and zero products of self-adjoint elements in \mathcal{A} then $\theta = \theta(1)J$ for a Jordan $*$ -homomorphism J from \mathcal{A} into \mathcal{B}^{**} . In [6], the involution preserving assumption is successfully removed. Modifying the arguments in [6], we will further relax the condition that the C^* -algebras are unital in this

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note. In particular, we show that \mathcal{A} and \mathcal{B} are isomorphic as Jordan algebras if θ is bijective and sends self-adjoint elements with zero products in \mathcal{A} to elements (not necessarily self-adjoint, though) with zero products in \mathcal{B} . They are isomorphic as C^* -algebras if θ is bijective and preserves the full zero product structure.

2. RESULTS

In the following, \mathcal{A}, \mathcal{B} are always C^* -algebras not necessarily with identities. \mathcal{A}_{sa} denotes the (real) Jordan-Banach algebra consisting of all self-adjoint elements of \mathcal{A} .

Recall that a linear map J between two algebras is said to be a *Jordan homomorphism* if $J(xy + yx) = J(x)J(y) + J(y)J(x)$ for all x, y . If the underlying field has characteristic not 2, this condition is equivalent to that $J(x^2) = (Jx)^2$ for all x in the domain. We also have the identity $J(xyx) = J(x)J(y)J(x)$ for all x, y in this case.

Lemma 2.1. *Let $J : \mathcal{A}_{sa} \rightarrow \mathcal{B}$ be a bounded Jordan homomorphism. Then J sends zero products in \mathcal{A}_{sa} to zero products in \mathcal{B} .*

Proof. Let a, b be self-adjoint elements in \mathcal{A} and $ab = 0$. We want to prove that $J(a)J(b) = 0$. Without loss of generality, we can assume that $a \geq 0$. Let a' in \mathcal{A}_{sa} satisfy that $a'^2 = a$. We have $a'b = 0$. By the identities $0 = J(a'ba') = J(a')J(b)J(a')$ and $0 = J(a'b + ba') = J(a')J(b) + J(b)J(a')$, we have $J(a)J(b) = J(a'^2)J(b) = J(a')^2J(b) = 0$. \square

Recall that when we consider \mathcal{A}^{**} as the enveloping W^* -algebra of \mathcal{A} , the multiplier algebra $M(\mathcal{A})$ of \mathcal{A} is the C^* -subalgebra of \mathcal{A}^{**} ,

$$M(\mathcal{A}) = \{x \in \mathcal{A}^{**} : x\mathcal{A} \subseteq \mathcal{A} \text{ and } \mathcal{A}x \subseteq \mathcal{A}\}.$$

Elements in $M(\mathcal{A})_{sa}$ can be approximated by both monotone increasing and decreasing bounded nets from $\tilde{\mathcal{A}}_{sa} = \mathcal{A}_{sa} \oplus \mathbb{R}1$ (see, e.g., [5]). In case \mathcal{A} is unital, $M(\mathcal{A}) = \mathcal{A}$.

Lemma 2.2. *Let $\theta : \mathcal{A}_{sa} \rightarrow \mathcal{B}$ be a bounded linear map sending zero products in \mathcal{A}_{sa} to zero products in \mathcal{B} . Then the restriction of θ^{**} induces a bounded linear map, denoted again by θ , from $M(\mathcal{A})_{sa}$ into \mathcal{B}^{**} , which sends zero products in $M(\mathcal{A})_{sa}$ to zero products in \mathcal{B}^{**} .*

Proof. First we consider the case $b \in \mathcal{A}_{sa}$, and p is an open projection in \mathcal{A}^{**} such that $pb = 0$. For any self-adjoint element c in the hereditary C^* -subalgebra $h(p) = p\mathcal{A}^{**}p \cap \mathcal{A}$ of \mathcal{A} , we have $cb = 0$ and thus $\theta(c)\theta(b) = 0$. By the weak* continuity of θ^{**} , we have $\theta^{**}(p\mathcal{A}_{sa}^{**}p)\theta(b) = 0$. In particular, $\theta^{**}(p)\theta(b) = 0$.

Let a, b be self-adjoint elements in $M(\mathcal{A})$ with $ab = 0$. We want to prove that $\theta(a)\theta(b) = 0$. Without loss of generality, we can assume both a, b are positive. Let $0 \leq a_\alpha + \lambda_\alpha \uparrow a$ be a monotone increasing net from $\tilde{\mathcal{A}}_{sa}$. Since $0 \leq b(a_\alpha + \lambda_\alpha)b \uparrow bab = 0$, we have $(a_\alpha + \lambda_\alpha)b = 0$ for all α . Similarly, there is a monotone increasing net $0 \leq b_\beta + s_\beta \uparrow b$ from $\tilde{\mathcal{A}}_{sa}$ such that $(a_\alpha + \lambda_\alpha)(b_\beta + s_\beta) = 0$ for all β . We can assume the real scalar $\lambda_\alpha \neq 0$. Then $s_\beta = 0$ for all β . In particular, we see that a_α commutes with all b_β . In the abelian C^* -subalgebra of $M(\mathcal{A})$ generated by a_α , b_β and 1 , we see that $a_\alpha + \lambda_\alpha$ can be approximated in norm by finite real linear combinations of open projections disjoint from b_β . By the first paragraph, we have $\theta(a_\alpha + \lambda_\alpha)\theta(b_\beta) = 0$.

By the weak* continuity of θ^{**} again, we see that $\theta(a_\alpha + \lambda_\alpha)\theta(b) = \lim_\beta \theta(a_\alpha + \lambda_\alpha)\theta(b_\beta) = 0$ for each α , and then $\theta(a)\theta(b) = \lim_\alpha \theta(a_\alpha + \lambda_\alpha)\theta(b) = 0$. \square

With Lemma 2.2, results in [6] concerning zero product preservers of unital C*-algebras can be extended easily to the non-unital case. We restate [6, Lemmas 4.4 and 4.5] below, but now here \mathcal{A} does not necessarily have an identity.

Lemma 2.3. *Let $\theta : \mathcal{A} \rightarrow \mathcal{B}$ be a bounded linear map sending zero products in \mathcal{A}_{sa} to zero products in \mathcal{B} . For any a in $M(\mathcal{A})$, we have*

- (i) $\theta(1)\theta(a) = \theta(a)\theta(1)$,
- (ii) $\theta(1)\theta(a^2) = (\theta(a))^2$.

If $\theta(1)$ is invertible then $\theta = \theta(1)J$ for a bounded Jordan homomorphism J from \mathcal{A} into \mathcal{B} .

Theorem 2.4. *Two C*-algebras \mathcal{A} and \mathcal{B} are isomorphic as Jordan algebras if and only if there is a bounded bijective linear map θ between them sending zero products in \mathcal{A}_{sa} to zero products in \mathcal{B} . If θ is just surjective, then \mathcal{B} is isomorphic to the C*-algebra $\mathcal{A}/\ker\theta$ as Jordan algebras.*

Proof. One way follows from Lemma 2.1. Conversely, suppose $\theta(\mathcal{A}) = \mathcal{B}$. Since $\theta(1)\theta(a^2) = \theta(a)^2$ for all a in \mathcal{A} and $\mathcal{B} = \mathcal{B}^2$, we have $\theta(1)\mathcal{B} = \mathcal{B}$. Thus, the central element $\theta(1)$ is invertible. Lemma 2.3 applies, by noting that closed Jordan ideals of C*-algebras are two-sided ideals [7]. \square

In case θ preserves all zero products in \mathcal{A} , we have the following non-unital version of [6, Theorem 4.11].

Theorem 2.5. *Let θ be a surjective bounded linear map from a C*-algebra \mathcal{A} onto a C*-algebra \mathcal{B} . Suppose that $\theta(a)\theta(b) = 0$ for all $a, b \in \mathcal{A}$ with $ab = 0$. Then $\theta(1)$ is a central element and invertible in $M(\mathcal{B})$. Moreover, $\theta = \theta(1)\varphi$ for a surjective algebra homomorphism φ from \mathcal{A} onto \mathcal{B} .*

Proof. First, we have already seen in the proof of Theorem 2.4 that $\theta(1)$ is a central element and invertible in $M(\mathcal{B})$. Second, we observe that to utilize the results [6, Theorems 4.12 and 4.13] of Brešar [4], and [6, Lemma 4.14] of Akemann and Pedersen [2], one does not need to assume \mathcal{A} or \mathcal{B} is unital. Together with our new Theorem 2.4, which is a non-unital version of [6, Theorem 4.6], we can now make use of the same proof of [6, Theorem 4.11] to establish the assertion. \square

Motivated by the theory of Banach lattices (see, e.g., [1]), we call two C*-algebras being *d-isomorphic* if there is a bounded bijective linear map between them sending zero-products to zero-products. We end this note with the following

Corollary 2.6. *Two C*-algebras are d-isomorphic if and only if they are *-isomorphic.*

Proof. The conclusion follows from Theorem 2.5 and a result of Sakai [8, Theorem 4.1.20] stating that two algebraic isomorphic C*-algebras are indeed *-isomorphic. \square

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